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(Automorphic forms, trace formulas and  
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# LIFTING OF PAIRS OF ELLIPTIC MODULAR FORMS TO SIEGEL MODULAR FORMS OF HALF-INTEGRAL WEIGHT OF DEGREE TWO

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## 1. INTRODUCTION

The aim of this exposition is to explain our recent work on a certain lifting from pairs of two elliptic modular forms to Siegel modular forms of half-integral weight of degree two [H11a, H11b]. The existence of this lifting has been conjectured by Ibukiyama and the author [HI05] as follows.

**Conjecture 1** ([HI05]). *Let  $k$  be an integer. Let  $f \in S_{2k-2}(SL_2(\mathbb{Z}))$ ,  $g \in S_{2k-4}(SL_2(\mathbb{Z}))$  be elliptic modular forms of weight  $2k-2$  and  $2k-4$ , respectively. We assume that  $f$  and  $g$  are normalized Hecke eigenforms.*

*Then, there exists  $F \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$  such that  $F$  is an eigenform for any Hecke operators, and the  $L$ -function of  $F$  satisfies the identity*

$$L(s, F) = L(s, f) L(s-1, g).$$

Here  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$  is a generalization of the Kohnen plus space for Siegel modular forms of degree two, and where  $L(s, F)$  denotes the  $L$ -function of  $F \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ , and where  $L(s, f)$  and  $L(s, g)$  denote the usual  $L$ -functions of  $f$  and  $g$ , respectively. We remark that the  $L$ -function of modular forms of half-integral weight was first introduced by Shimura [Sh73], and was generalized by Zhuravlev [Zh84] for Siegel modular forms of half-integral weight. The above  $L$ -function  $L(s, F)$  contains the Euler 2-factor which is introduced in [HI05] for  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ . Some numerical examples of Euler factors have supported the above conjecture.

The following theorem is the main result of this exposition.

**Theorem 2.** *Let  $k$  be an even integer. Let  $f$  and  $g$  be as in the above conjecture. Then we obtain a Siegel modular form  $\mathcal{F}_{f,g} \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$  from the pair of  $f$  and  $g$ . And if  $\mathcal{F}_{f,g}$  does not vanish identically, then  $\mathcal{F}_{f,g}$  satisfies the properties in the above conjecture.*

## 2. CONSTRUCTION OF $\mathcal{F}_{f,g}$

From now on we assume that  $k$  is an even integer, and we assume that  $f \in S_{2k-2}(SL_2(\mathbb{Z}))$  and  $g \in S_{2k-4}(SL_2(\mathbb{Z}))$  are normalized Hecke eigenforms, namely, the first Fourier coefficients of  $f$  and  $g$  are both 1.

The purpose of this section is to explain the construction of the Siegel modular form  $\mathcal{F}_{f,g} \in S_{k-1/2}^+(\Gamma_0^{(2)}(4))$  which will satisfy the properties in Theorem 2. This construction was suggested by Prof. T. Ikeda to the author in 2001 at the Hakuba conference.

We denote by  $S_k(\mathrm{Sp}_4(\mathbb{Z}))$  the space of Siegel cusp forms of weight  $k$  of degree 4, and denote by  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(3)}(4))$  the generalized plus space of weight  $k - \frac{1}{2}$  of degree 3 (cf. [Ib92]), which is a certain subspace of Siegel modular forms of weight  $k - \frac{1}{2}$  of degree three.

Let  $I(g) \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$  be the Duke-Imamoglu-Ikeda lift of  $g$ . We consider a Fourier-Jacobi expansion of  $I(g)$ :

$$I(g) \left( \begin{pmatrix} \tau_3 & z \\ z & \omega_1 \end{pmatrix} \right) = \sum_{n>0} \Psi_n(\tau_3, z) e^{2\pi\sqrt{-1}n\omega_1},$$

where  $\tau_3 \in \mathfrak{H}_3$ ,  $\omega_1 \in \mathfrak{H}_1$  and  $z \in M_{3,1}(\mathbb{C})$ . Here we denote by  $\mathfrak{H}_n$  the Siegel upper half space of degree  $n$  and by  $M_{n,m}(K)$  the matrices of size  $n$  by  $m$  with entries in a commutative ring  $K$ . We remark that the form  $\Psi_n$  is a Jacobi form of index  $n$  of weight  $k$  of degree 3. For the definition of Jacobi forms of higher degree the reader is referred to [Zi89].

By the isomorphism between the generalized plus space and the space of Jacobi forms of index one (cf. [Ib92]), there exists  $G \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(3)}(4))$  which corresponds to the Jacobi form  $\Psi_1$  of index 1.

We consider the following expansion of a pullback of  $G$ :

$$G \left( \begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix} \right) = \sum_h \mathcal{F}_{h,g}(\tau_2) h(\omega_1),$$

where  $\tau_2 \in \mathfrak{H}_2$  and  $\omega_1 \in \mathfrak{H}_1$ , and where  $h$  runs over all elements in a basis which consists of Hecke eigenforms in the Kohnen plus space  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(1)}(4))$ . We see that  $\mathcal{F}_{h,g}$  belongs to  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$  for any  $h$ . In particular there exists  $\hat{f} \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(1)}(4))$  which corresponds to  $f$  by the Shimura correspondence. Therefore, we define

$$\mathcal{F}_{f,g} := \mathcal{F}_{\hat{f},g},$$

then the form  $\mathcal{F}_{f,g}$  belongs to  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ .

To complete the proof of Theorem 2 we need to show that the form  $\mathcal{F}_{f,g}$  is an eigenform for any Hecke operators under the assumption  $\mathcal{F}_{f,g} \not\equiv 0$ , and we need to show that the L-function  $L(s, \mathcal{F}_{f,g})$  of  $\mathcal{F}_{f,g}$  coincides with  $L(s-1, g)L(s, f)$ . We shall show these facts in the following sections.

### 3. FOURIER-JACOBI EXPANSION OF $G$

Let  $G \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(3)}(4))$  be the form defined in the previous section. We consider a Fourier-Jacobi expansion of  $G$ :

$$G\left(\begin{pmatrix} \tau_2 & z \\ z & \omega_1 \end{pmatrix}\right) = \sum_{m>0} \phi_m(\tau_2, z) e^{2\pi\sqrt{-1}m\omega_1},$$

where  $\tau_2 \in \mathfrak{H}_2$ ,  $\omega_1 \in \mathfrak{H}_1$  and  $z \in M_{2,1}(\mathbb{C})$ .

The purpose of this section is to give a certain relation among  $\phi_m$ , which plays an important role in the proof of Theorem 2. We remark that the form  $\phi_m$  is a Jacobi form of index  $m$  of weight  $k - \frac{1}{2}$ . Here, Jacobi forms of index  $m$  of weight  $k - \frac{1}{2}$  of degree 2 are holomorphic functions on  $\mathfrak{H}_2 \times M_{2,1}(\mathbb{C})$  which satisfy a certain transformation formula (cf. [H11a].) We denote by  $J_{k-\frac{1}{2},m}^{(2)}$  the space of Jacobi forms of index  $m$  of weight  $k - \frac{1}{2}$  of degree 2.

For any prime  $p$  we define two maps  $V_{1,p}^{(2)}$  and  $V_{2,p}^{(2)}$  which are maps from  $J_{k-\frac{1}{2},m}^{(2)*}$  to  $J_{k-\frac{1}{2},mp^2}^{(2)}$  (the reader is referred to [H11a] for the precise definition of  $V_{i,p}^{(2)}$ ), where  $J_{k-\frac{1}{2},m}^{(2)*}$  is a certain subspace of  $J_{k-\frac{1}{2},m}^{(2)}$ . These  $V_{1,p}^{(2)}$  and  $V_{2,p}^{(2)}$  are certain generalizations of the  $V_l$ -operator in [EZ85, p.43]. We also define the map  $U_l : J_{k-\frac{1}{2},m}^{(2)} \rightarrow J_{k-\frac{1}{2},ml^2}^{(2)}$  defined by  $(\phi|U_l)(\tau_2, z) := \phi(\tau_2, lz)$  for any  $\phi \in J_{k-\frac{1}{2},m}^{(2)}$ .

Now we obtain the following relations.

**Theorem 3.** *Let  $\phi_m$  be as above. For any prime  $p$  we have*

$$\phi_m|V_{1,p}^{(2)} = pb(p)\phi_m|U_p + \phi_{mp^2} + \left(\frac{-m}{p}\right)p^{k-2}\phi_m|U_p + p^{2k-3}\phi_{\frac{m}{p^2}}|U_{p^2}$$

and

$$\begin{aligned} \phi_m|V_{2,p}^{(2)} &= b(p)\left(\phi_{mp^2} + \left(\frac{-m}{p}\right)p^{k-2}\phi_m|U_p + p^{2k-3}\phi_{\frac{m}{p^2}}|U_{p^2}\right) \\ &\quad + (p^{2k-4} - p^{2k-6})\phi_m|U_p, \end{aligned}$$

where  $b(p)$  is the  $p$ -th Fourier coefficient of  $g$ .

We remark that the above identities can be translated to relations among Fourier coefficients of  $G$ . The outline of the proof of Theorem 3 will be given in Section 5. We also remark that the above identities can be regarded as certain generalizations of the Maass relation for Siegel modular forms of half-integral weight of degree three.

#### 4. PROOF OF THEOREM 2

In this section we assume Theorem 3 and shall prove Theorem 2. We use the same symbols in Section 2, 3.

For a prime  $p$ , let  $X_1(p)$  and  $X_2(p)$  be Hecke operators which generate the local Hecke ring acting on  $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ . For the precise definition of these Hecke operators the reader is referred to [HI05, p.513]. By the virtue of the definitions of  $V_{1,p}^{(2)}$  and  $V_{2,p}^{(2)}$ , we can obtain

$$\left(\phi|V_{1,p}^{(2)}\right)(\tau_2, 0) = p^{-k+\frac{7}{2}}\phi(\tau_2, 0)|X_1(p)$$

and

$$\left(\phi|V_{2,p}^{(2)}\right)(\tau_2, 0) = \phi(\tau_2, 0)|X_2(p)$$

for any  $\phi \in J_{k-\frac{1}{2},m}^{(2)*}$ , and where we regard  $\phi(\tau_2, 0)$  as a function of  $\tau_2 \in \mathfrak{H}_2$ .

Therefore, due to Theorem 3, we obtain

$$\begin{aligned} & p^{-k+\frac{7}{2}}G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|X_1(p) \\ &= \sum_m p^{-k+\frac{7}{2}}((\phi_m(\tau_2, 0))|X_1(p))e^{2\pi\sqrt{-1}m\omega_1} \\ &= \sum_m \left((\phi_m|V_{1,p}^{(2)})(\tau_2, 0)\right)e^{2\pi\sqrt{-1}m\omega_2} \\ &= \sum_m \{pb(p)\phi_m(\tau_2, 0) + \phi_{mp^2}(\tau_2, 0) \\ &\quad + \left(\frac{-m}{p}\right)p^{k-2}\phi_m(\tau_2, 0) + p^{2k-3}\phi_{\frac{m}{p^2}}(\tau_2, 0)\}e^{2\pi\sqrt{-1}m\omega_1} \\ &= pb(p)G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) + G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|T_1(p^2). \end{aligned}$$

Similarly we have

$$\begin{aligned} & G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|X_2(p) \\ &= b(p)G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|T_1(p^2) + (p^{2k-4} - p^{2k-6})G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right). \end{aligned}$$

Because  $G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \sum_h \mathcal{F}_{h,g}(\tau_2) h(\omega_1)$  and because of the above identities, we obtain

$$p^{-k+\frac{7}{2}} \mathcal{F}_{f,g}|X_1(p) = (pb(p) + a(p)) \mathcal{F}_{f,g}$$

and

$$\mathcal{F}_{f,g}|X_2(p) = (a(p)b(p) + p^{2k-4} - p^{2k-6}) \mathcal{F}_{f,g},$$

where  $a(p)$  is the  $p$ -th Fourier coefficient of  $f$ . Hence  $\mathcal{F}_{f,g}$  is an eigenform for any Hecke operators. Moreover the eigenvalues of  $\mathcal{F}_{f,g}$  follow from the above identities. Let  $\lambda(p) = p^{k-\frac{7}{2}}(pb(p) + a(p))$  and  $\omega(p) = a(p)b(p) + p^{2k-4} - p^{2k-6}$  be the eigenvalues of  $\mathcal{F}_{f,g}$  with respect to  $X_1(p)$  and  $X_2(p)$ , respectively. Then we obtain

$$\begin{aligned} L(s, \mathcal{F}_{f,g}) &= \prod_p \left( 1 - \lambda(p)p^{-k+\frac{7}{2}-s} + (p\omega(p) + p^{2k-5}(1+p^2))p^{-2s} \right. \\ &\quad \left. - \lambda(p)p^{k+\frac{1}{2}-3s} + p^{4k-6-4s} \right)^{-1} \\ &= L(s, f) L(s-1, g). \end{aligned}$$

□

## 5. PROOF OF THEOREM 3

In this section we shall give the outline of the proof of Theorem 3. The steps for the proof of Theorem 3 are as follows.

- (1) By the virtue of the Duke-Imamoglu-Ikeda lift, it is enough to show Theorem 3 for the case of generalized Cohen-Eisenstein series. Here, the generalized Cohen-Eisenstein series are certain Siegel modular forms of half-integral weight, which are not cusp forms (cf. [Co75] for degree one, and [Ar98] for general degree).
- (2) Show certain linear isomorphisms between the space of certain Jacobi forms of half-integral weight of integer index and the space of Jacobi forms of integral weight of matrix index.
- (3) Show a compatibility between the linear isomorphisms in (2) and certain operators which shift the indices of Jacobi forms.
- (4) We need a certain relation between Fourier-Jacobi coefficients of Siegel-Eisenstein series and Jacobi-Eisenstein series of matrix index shown by S. Boecherer [Bo83].
- (5) Calculate the action of shift operators on Jacobi-Eisenstein series of matrix index explicitly.
- (6) Show Theorem 3 for the case of generalized Cohen-Eisenstein series by using (2), (3), (4) and (5).

In this section we will explain the above steps more precisely.

**5.1. Generalized Cohen-Eisenstein series.** Let  $k'$  be an even integer. We denote by  $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$  the generalized Cohen-Eisenstein series of weight  $k' - \frac{1}{2}$  of degree 3, which is a certain Siegel modular form of weight  $k' - \frac{1}{2}$  of degree 3 (cf. [Co75, Ar98]). More precisely, the form  $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$  corresponds to the Jacobi-Eisenstein series  $E_{k',1}^{(3)}$  of index 1 of weight  $k'$  of degree 3 by the isomorphism between Siegel modular forms of half-integral weight and Jacobi forms of index one (cf. [EZ85, Ib92].)

By the virtue of the Duke-Imamoglu-Ikeda lift, the Fourier coefficients of  $G$  satisfy the similar relation which the Fourier coefficients of  $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$  satisfy. Hence, it is enough to show Theorem 3 for the case of  $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$  for sufficiently many  $k'$ .

We take a Fourier-Jacobi expansion of  $\mathcal{H}_{k-\frac{1}{2}}^{(3)}$ :

$$\mathcal{H}_{k-\frac{1}{2}}^{(3)} \left( \begin{pmatrix} \tau_2 & z \\ z & \omega_1 \end{pmatrix} \right) = \sum_{m \geq 0} e_{k-\frac{1}{2},m}^{(2)}(\tau_2, z) e^{2\pi\sqrt{-1}m\omega_1},$$

where  $\tau_2 \in \mathfrak{H}_2$ ,  $\omega_1 \in \mathfrak{H}_1$  and  $z \in M_{2,1}(\mathbb{C})$ . The form  $e_{k-\frac{1}{2},m}^{(2)}$  is a Jacobi form of index  $m$  of weight  $k - \frac{1}{2}$  of degree 2, which belongs to  $J_{k-\frac{1}{2},m}^{(2)*}$ .

We need to show the following theorem.

**Theorem 4.** *Let  $k > 5$  be an even integer and let  $e_{k-\frac{1}{2},m}^{(2)}$  be as above. For any prime  $p$  we have*

$$\begin{aligned} e_{k-\frac{1}{2},m}^{(2)}|V_{1,p}^{(2)} &= p(1+p^{2k-5})e_{k-\frac{1}{2},m}^{(2)}|U_p + e_{k-\frac{1}{2},mp^2}^{(2)} \\ &\quad + \left(\frac{-m}{p}\right)p^{k-2}e_{k-\frac{1}{2},m}^{(2)}|U_p + p^{2k-3}e_{k-\frac{1}{2},\frac{m}{p^2}}^{(2)}|U_{p^2} \end{aligned}$$

and

$$\begin{aligned} e_{k-\frac{1}{2},m}^{(2)}|V_{2,p}^{(2)} &= (1+p^{2k-5}) \left\{ e_{k-\frac{1}{2},mp^2}^{(2)} + \left(\frac{-m}{p}\right)p^{k-2}e_{k-\frac{1}{2},m}^{(2)}|U_p \right. \\ &\quad \left. + p^{2k-3}e_{k-\frac{1}{2},\frac{m}{p^2}}^{(2)}|U_{p^2} \right\} + (p^{2k-4} - p^{2k-6})e_{k-\frac{1}{2},m}^{(2)}|U_p. \end{aligned}$$

Theorem 3 follows from this theorem.

**5.2. An isomorphism between two spaces of Jacobi forms.** Let  $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$  be the space of Siegel modular forms of weight  $k$  of even degree  $n+2$ , and let  $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$  be the subspace of  $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$  which consists of all Duke-Imamoglu-Ikeda lifts in  $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ . We remark that the subspace  $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$  contains the Siegel-Eisenstein series.

Let  $J_{k,1}^{(n+1)*}$  be the subspace of  $J_{k,1}^{(n+1)}$  which consists of all Jacobi forms of index 1 obtained by the Fourier-Jacobi expansion of the Siegel modular forms in  $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ . We denote by  $M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$  the subspace of the generalized plus space  $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$  of degree  $n+1$  which corresponds to  $J_{k,1}^{(n+1)*}$  by the isomorphism between two spaces  $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$  and  $J_{k,1}^{(n+1)}$ . Now, we have the following diagram.

$$\begin{array}{ccc} M_k(\mathrm{Sp}_{n+2}(\mathbb{Z})) & \supset & M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z})) \\ \downarrow & & \downarrow \\ J_{k,1}^{(n+1)} & \supset & J_{k,1}^{(n+1)*} \cong M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \end{array}$$

Let  $\phi_1 \in J_{k,1}^{(n+1)}$ . We regard the function  $\phi_1(\tau_{n+1}, z) e^{2\pi\sqrt{-1}\omega_1}$  as a function on  $\begin{pmatrix} \tau_{n+1} & z \\ t_z & \omega_1 \end{pmatrix} \in \mathfrak{H}_{n+2}$ , where  $\tau_{n+1} \in \mathfrak{H}_{n+1}$ ,  $\omega_1 \in \mathfrak{H}_1$  and  $z \in M_{n+1,1}(\mathbb{C})$ . We consider the expansion:

$$\phi_1(\tau_{n+1}, z) e^{2\pi\sqrt{-1}\omega_1} = \sum_{\substack{S \in \mathrm{Sym}_2^* \\ S = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}}} \phi_S(\tau_n, z') e^{2\pi\sqrt{-1}\mathrm{tr}(S\omega_2)}$$

where  $\mathrm{Sym}_2^*$  denotes the set of all half-integral symmetric matrices of size 2, and where  $\begin{pmatrix} \tau_{n+1} & z \\ t_z & \omega_1 \end{pmatrix} = \begin{pmatrix} \tau_n & z' \\ t_{z'} & \omega_2 \end{pmatrix} \in \mathfrak{H}_{n+2}$ ,  $\tau_n \in \mathfrak{H}_n$ ,  $\omega_2 \in \mathfrak{H}_2$  and  $z' \in M_{n,2}(\mathbb{C})$ . We remark that  $\phi_S \in J_{k,S}^{(n)}$ , where  $J_{k,S}^{(n)}$  is the space of Jacobi forms of weight  $k$  of index  $S$  of degree  $n$  (cf. [Zi89], [H11a]). We consider the map

$$\mathrm{FJ}_{1,S} : J_{k,1}^{(n+1)} \rightarrow J_{k,S}^{(n)} \quad \text{via} \quad \phi_1 \mapsto \phi_S.$$

We denote by  $J_{k,S}^{(n)*} \subset J_{k,S}^{(n)}$  the image of  $J_{k,1}^{(n+1)*}$  by the map  $\mathrm{FJ}_{1,S}$ . Namely, the Jacobi forms in  $J_{k,S}^{(n)*}$  are obtained by the Duke-Imamoglu-Ikeda lifting and by the Fourier-Jacobi expansion.

Let  $m$  be an integer such that  $-m \equiv 0, 1 \pmod{4}$ . We take a matrix  $\mathcal{M} \in \mathrm{Sym}_2^*$ , such that  $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$  and  $\det(2\mathcal{M}) = m$ .

**Lemma 5.** *The space  $J_{k,\mathcal{M}}^{(n)*}$  is linearly isomorphic to the space  $J_{k-\frac{1}{2},m}^{(n)*}$  and this isomorphism is given by a correspondence between Fourier coefficients. We denote by  $\iota_{\mathcal{M}}$  this map. Moreover, the following diagram*



is commutative.

$$\begin{array}{ccc}
 J_{k,1}^{(n+1)*} & \longrightarrow & M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \\
 FJ_{1,\mathcal{M}} \downarrow & & \downarrow \\
 J_{k,\mathcal{M}}^{(n)*} & \xrightarrow{\iota_{\mathcal{M}}} & J_{k-\frac{1}{2},m}^{(n)*}
 \end{array}$$

where the map of the down arrow in the right hand side is given by the Fourier-Jacobi expansion.

Due to the above lemma the  $m$ -th Fourier-Jacobi coefficient  $e_{k-\frac{1}{2},m}^{(n)}$  of  $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$  corresponds to a certain Jacobi form  $e_{k,\mathcal{M}}^{(n)}$  of index  $\mathcal{M}$ . Here  $e_{k-\frac{1}{2},m}^{(n)} \in J_{k-\frac{1}{2},m}^{(n)*}$ ,  $e_{k,\mathcal{M}}^{(n)} \in J_{k,\mathcal{M}}^{(n)*}$  and  $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)} \in M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$ . In particular, we can say that the form  $e_{k,\mathcal{M}}^{(n)}$  is the  $\mathcal{M}$ -th Fourier-Jacobi coefficient of the Siegel-Eisenstein series of weight  $k$  of degree  $n+2$  (see §5.4 below).

Now we want to translate the relation among  $e_{k-\frac{1}{2},m}^{(2)}$  in Theorem 4 to the relation among  $e_{k,\mathcal{M}}^{(2)}$ . We need to show a compatibility between the above isomorphism  $\iota_{\mathcal{M}} : J_{k,\mathcal{M}}^{(n)*} \xrightarrow{\sim} J_{k-\frac{1}{2},m}^{(n)*}$  and certain operators acting on each space. In the next subsection we will explain this compatibility.

**5.3. Compatibility between isomorphisms of Jacobi forms and certain operators.** We can define certain operators

$$V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)} : J_{k,\mathcal{M}}^{(n)} \rightarrow J_{k,\mathcal{M}[(\begin{smallmatrix} p & 1 \end{smallmatrix})]}^{(n)} \quad (i = 1, \dots, n).$$

Namely, the operator  $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$  changes the matrix of the index of Jacobi forms. The operator  $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$  is a generalization of  $V_l$ -operator in [EZ85, p.43]. For the precise definition of  $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$  the reader is referred to [H11a].

**Lemma 6.** *Let  $\mathcal{M}$  be a matrix in Lemma 5. Then, the form*

$$\iota_{\mathcal{M}[(\begin{smallmatrix} p & 1 \end{smallmatrix})]}(e_{k,\mathcal{M}}^{(n)} | V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)})$$

*is the same to  $e_{k-\frac{1}{2},m}^{(n)} | V_{i,p}^{(n)}$  up to constant. Namely, the isomorphism  $\iota_{\mathcal{M}} : J_{k,\mathcal{M}}^{(n)*} \xrightarrow{\sim} J_{k-\frac{1}{2},m}^{(n)*}$  and the operators  $V_{i,p}^{(n)}$ ,  $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$  are compatible.*

We remark that the form  $e_{k,\mathcal{M}}^{(n)}|V_{i,(p \ 1)}^{(n)} \in J_{k,\mathcal{M}[(p \ 1)]}^{(n)}$  may not be in  $J_{k,\mathcal{M}[(p \ 1)]}^{(n)*}$ , but we can extend the map  $\iota_{\mathcal{M}[(p \ 1)]}$  to  $J_{k,\mathcal{M}[(p \ 1)]}^{(n)}$ .

Due to Lemma 5 and 6, we can translate the relation among  $e_{k-\frac{1}{2},m}^{(2)}$  ( $m \in \mathbb{Z}$ ) in Theorem 4 to the relation among  $e_{k,\mathcal{M}}^{(2)}$  ( $\mathcal{M} \in \text{Sym}_2^*$ ). Hence, it is enough to show the same relation in Theorem 4 for  $e_{k,\mathcal{M}}^{(2)}$ . For the calculation of  $e_{k,\mathcal{M}}^{(2)}|V_{i,(p \ 1)}^{(2)}$ , we need relations between  $e_{k,\mathcal{M}}^{(2)}$  and Jacobi-Eisenstein series. We will explain this relation in the next subsection.

**5.4. Fourier-Jacobi coefficients of Siegel-Eisenstein series.** We denote by  $E_k^{(n+2)}$  the Siegel-Eisenstein series of weight  $k$  of degree  $n+2$ . We take the Fourier-Jacobi expansion:

$$E_k^{(n+2)}\left(\begin{pmatrix} \tau_n & z \\ t & \omega_2 \end{pmatrix}\right) = \sum_{\mathcal{M} \in \text{Sym}_2^*} e_{k,\mathcal{M}}^{(n)}(\tau_n, z) e^{2\pi\sqrt{-1}\text{tr}(\mathcal{M}\omega_2)},$$

where  $\tau_n \in \mathfrak{H}_n$ ,  $\omega_2 \in \mathfrak{H}_2$  and  $z \in M_{n,2}(\mathbb{C})$ .

We denote by  $\text{Sym}_2^+$  the set of all positive-definite half-integral symmetric matrices of size 2. We let  $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ . We put  $m = \det(2\mathcal{M})$ , and let  $D_0$  be the discriminant of  $\mathbb{Q}(\sqrt{-m})$ , and put  $f = \sqrt{-\frac{m}{D_0}}$ . We remark that  $f$  is a natural number.

We set  $g_k(m) := \sum_{d|f} \mu(d) h_{k-\frac{1}{2}}\left(\frac{m}{d^2}\right)$ , where  $\mu(d)$  is the Möbius function and  $h_{k-\frac{1}{2}}(d)$  denotes the  $d$ -th Fourier coefficient of the Cohen-Eisenstein series of weight  $k - \frac{1}{2}$  (cf. [Co75].)

We denote by  $E_{k,\mathcal{S}}^{(n)}$  the Jacobi-Eisenstein series of index  $\mathcal{S} \in \text{Sym}_2^+$  of weight  $k$  of degree  $n$  (cf. [Zi89].) The following proposition follows from [Bo83, Satz 7].

**Proposition 7.** For  $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$  we put  $m = \det(2\mathcal{M})$ . Let  $D_0, f$  be as above. If  $k > n + 3$ , then

$$e_{k,\mathcal{M}}^{(n)}(\tau, z) = \sum_{d|f} g_k\left(\frac{m}{d^2}\right) E_{k,\mathcal{M}[\iota W_d^{-1}]}^{(n)}(\tau, z W_d),$$

where we chose a matrix  $W_d \in GL_2(\mathbb{Q}) \cap M_{2,2}(\mathbb{Z})$  for each  $d$  which satisfies the conditions  $\det(W_d) = d$  and  $W_d^{-1} \mathcal{M}^t W_d^{-1} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ .

The above summation is well-defined, namely it does not depend on the choice of matrix  $W_d$ .

Hence, relations among  $e_{k,\mathcal{M}}^{(2)}$  is translated to the relations among  $E_{k,\mathcal{M}}^{(2)}$ . Namely, for the calculation of  $e_{k,\mathcal{M}}^{(2)}|V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$  it is enough to calculate  $E_{k,\mathcal{M}[{}^tW_d^{-1}]}^{(2)}|V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$ , because  $V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$  is a linear map.

**5.5. Jacobi-Eisenstein series  $E_{k,\mathcal{M}}^{(2)}$  and operators  $V^{(2)}$ .** Let  $E_{k,\mathcal{M}}^{(2)}$  be the Jacobi-Eisenstein series of weight  $k$  of index  $\mathcal{M} \in \text{Sym}_2^+$  of degree 2, which is a holomorphic function on  $\mathfrak{H}_2 \times \mathbb{C}^2$ .

**Proposition 8.** *We assume  $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$ . Then, the form  $E_{k,\mathcal{M}}^{(2)}|V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$  ( $i = 1, 2$ ) is written as a linear combination of three forms*

$$E_{k,\mathcal{M}}^{(2)}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}), \quad E_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}^{(2)}(\tau, z),$$

and

$$E_{k,\mathcal{M}[X^{-1}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}]}^{(2)}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} {}^tX \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}),$$

explicitly. Here, if  $p|f$ , then  $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$  is a  $2 \times 2$  matrix with an integer  $x$ , such that  $\mathcal{M}[X^{-1}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}] \in \text{Sym}_2^+$ , and if  $p \nmid f$ , then the third form of the above does not appear.

By Lemma 5, 6 and Proposition 7, 8, we obtain Theorem 4. Thus, we can conclude Theorem 3.

## 6. EXAMPLES OF NON-VANISHING OF $\mathcal{F}_{f,g}$

In Theorem 2 we need the assumption that  $\mathcal{F}_{f,g}$  does not identically vanish. In this final section we shall give some examples of non-vanishing of  $\mathcal{F}_{f,g}$ .

For even integer  $k \leq 24$ , the dimensions of the spaces  $S_{2k-4}(\text{SL}_2(\mathbb{Z}))$ ,  $S_{2k-2}(\text{SL}_2(\mathbb{Z}))$  and  $S_{k-1/2}^+(\Gamma_0^{(2)}(4))$  are given as follows.

$k$	2, 4, 6	8	10	12	14	16	18	20	22	24
$\dim S_{2k-4}(\text{SL}_2(\mathbb{Z}))$	0	1	1	1	2	2	2	3	3	3
$\dim S_{2k-2}(\text{SL}_2(\mathbb{Z}))$	0	0	1	1	1	2	2	2	3	3
$\dim S_{k-1/2}^+(\Gamma_0^{(2)}(4))$	0	0	1	1	2	4	4	6	9	10

We remark that the dimension formula for  $S_{k-1/2}^+(\Gamma_0^{(2)}(4))$  is given in [HI05].

**Lemma 9.** *Let  $k = 10, 12$  or  $14$ . Then, the form  $\mathcal{F}_{f,g} \in S_{k-1/2}^+(\Gamma_0^{(2)}(4))$  does not vanish identically for any normalized Hecke eigenforms  $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$  and  $g \in S_{2k-4}(\text{SL}_2(\mathbb{Z}))$ .*

*Proof.* We assume  $k = 10, 12$  or  $14$ . Then,  $\dim S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) = 1$ .

Let  $g \in S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$  be a normalized Hecke eigenform, and let  $G \in S_{k-1/2}^+(\Gamma_0^{(3)}(4))$  be the Siegel modular form of weight  $k - 1/2$  of degree 3 which is constructed from  $g$  in §2. Let  $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$  be a normalized Hecke eigenform, and let  $h \in S_{k-1/2}^+(\Gamma_0^{(1)}(4))$  be the modular form of weight  $k - \frac{1}{2}$  which corresponds to  $f$  by the Shimura correspondence. Here  $S_{k-1/2}^+(\Gamma_0^{(1)}(4))$  denotes the Kohnen plus space.

Because  $\dim S_{k-1/2}^+(\Gamma_0^{(1)}(4)) = 1$ , we have

$$G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \mathcal{F}_{f,g}(\tau_2)h(\omega_1),$$

where  $\tau_2 \in \mathfrak{H}_2$  and  $\omega_1 \in \mathfrak{H}_1$ . Hence, it is enough to show that  $G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_2 \end{pmatrix}\right)$  does not vanish identically. We take the expansion

$$G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \sum_{N \in \mathrm{Sym}_2^+, m \in \mathbb{Z}} K(N, m) e^{2\pi\sqrt{-1}\mathrm{tr}(N\tau_2)} e^{2\pi\sqrt{-1}m\omega_2},$$

and we take the Fourier expansion of  $G$ :

$$G(Z) = \sum_{M \in \mathrm{Sym}_3^+} C(M) e^{2\pi\sqrt{-1}\mathrm{tr}(MZ)}.$$

Then

$$K(N, m) = \sum_{\substack{l \in M_{2,1}(\mathbb{Z}) \\ 4Nm - l^t l > 0}} C\left(\begin{pmatrix} N & \frac{1}{2}l \\ \frac{1}{2}l^t & m \end{pmatrix}\right).$$

The Fourier coefficients of  $G$  can be calculated by the formula of the Fourier coefficients of the Duke-Imamoglu-Ikeda lift. By numerical calculations we obtain  $K\left(\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, 3\right) \neq 0$ . Therefore, we conclude  $\mathcal{F}_{f,g} \neq 0$ .  $\square$

## REFERENCES

- [Ar98] T. Arakawa: Köcher-Maass Dirichlet Series Corresponding to Jacobi forms and Cohen Eisenstein Series, *Comment. Math. Univ. St. Paul.* **47** No.1 (1998), 93–122.
- [Bo83] S. Boecherer: Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, *Math. Z.* **183** (1983), 21–46
- [Co75] H. Cohen: Sums involving the values at negative integers of  $L$ -functions of quadratic characters, *Math. Ann.* **217** (1975), 171–185.
- [EZ85] M. Eichler and D. Zagier: Theory of Jacobi Forms, Progress in Math. **55**, Birkhäuser, Boston-Basel-Stuttgart, (1985).

- [HI05] S. Hayashida and T. Ibukiyama: Siegel modular forms of half integral weights and a lifting conjecture, *Journal of Kyoto Univ*, **45** No.3 (2005), 489-530.
- [H11a] S. Hayashida: A Maass relation for generalized Cohen-Eisenstein series of degree two and of degree three, *in preparation*.
- [H11b] S. Hayashida: On the lifting of pairs of elliptic modular forms to Siegel modular forms of half-integral weight of degree two, *in preparation*.
- [Ib92] T. Ibukiyama: On Jacobi forms and Siegel modular forms of half integral weights, *Comment. Math. Univ. St. Paul.* **41** No.2 (1992), 109–124.
- [Sh73] G. Shimura : On modular forms of half integral weight, *Ann. of Math* (2) **97** (1973), 440–481.
- [Zh84] V. G. Zhuravlev: Euler expansions of theta transforms of Siegel modular forms of half-integral weight and their analytic properties, *Math. sbornik.* **123** (165) (1984), 174–194.
- [Zi89] C. Ziegler : Jacobi forms of higher degree, *Abh. Math. Sem. Univ. Hamburg.* **59** (1989), 191–224.

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